

On the singularity formation, backwards heat
equation and renormalization for the
Navier-Stokes equations-A regularity result.

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Abstract

This article models the energy cascade singularity process for the incompressible Navier-Stokes equations. It is shown that the energy density transport in the context of singularity is equivalent to a backwards heat equation. It is then shown through a renormalization type argument that this process leads to a logical contradiction, which shows that such energy cascade singularity process does not exist in the Navier-Stokes equations. This will imply regularity for the Navier-Stokes equations as it is shown that the formation of singularity is developed through a gradient flux of the energy density always.

1 Introduction

The Navier-Stokes equations are supposed to model any kind of fluid flow, including turbulent ones. The essential problem with the equations is the nonlinearity of the convective term. The pressure gradient represents transport and the diffusive term represents loss of energy in the system. In other words, to understand the equations, one has to understand the nonlinear convection term. The Navier-Stokes equations can be written in vector form as follows

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} = \nu \Delta \vec{u} - \nabla p \quad (1)$$

where $\vec{u}(x, y, z, t) \in \mathbb{R}^3$ is the velocity field of the fluid and $p(x, y, z, t) \in \mathbb{R}$ is the scalar pressure field. The diffusive term depends on a constant $\nu > 0$. The convective term is $\vec{u} \cdot \nabla \vec{u}$. This term can be decomposed according to the vector calculus identities

$$\vec{u} \cdot \nabla \vec{u} = \vec{\omega} \times \vec{u} + \nabla \left(\frac{1}{2} \vec{u} \cdot \vec{u} \right) \quad (2)$$

Substituting this into the Navier-Stokes equations, one has

$$\frac{\partial \vec{u}}{\partial t} = -\vec{\omega} \times \vec{u} - \nabla \left(p + \frac{1}{2} \vec{u} \cdot \vec{u} \right) + \nu \Delta \vec{u} \quad (3)$$

where $\vec{\omega} = \nabla \times \vec{u}$ is the vorticity of the fluid flow. This formulation of the Navier-Stokes equations is known as the Lamb formulation [1]. In addition we assume that the fluid flow is incompressible, that is

$$\nabla \cdot \vec{u} = 0 \quad (4)$$

The system is assumed to have some smooth and square-integrable initial data $\vec{u}^0 = \vec{u}(x, y, z, 0)$ over the whole space \mathbb{R}^3 .

For the well-known facts of the Navier-Stokes system, we refer to [2]

2 The singularity process

One of the most celebrated results in the field of Navier-Stokes equations is the Beale-Kato-Majda theorem. It states that if there is a singularity forming in the system, then the following holds

$$\lim_{t \rightarrow T} \int_0^T \|\vec{\omega}(\cdot, s)\|_{L^\infty} ds = \infty \quad (5)$$

In other words, if there is a singularity forming, the vorticity is accumulating in some region of space quite rapidly. Suppose now that there is indeed a build-up of singularity in terms of vorticity, as prescribed in the Beale-Kato-Majda criterion. As stated earlier, the singularity is forming on the singular set, which can be 1-dimensional. Nevertheless, as shown in Caffarelli et al.[3], the singular set

can be covered with a minimal compact set $C \subset \mathbb{R}^3$ having a smooth surface ∂C . As the Navier-Stokes system is deterministic, the singular set for given initial data is definitely some set $S \subset C \subset \mathbb{R}^3$.

We now therefore assume a situation where the system is forming a singularity on some singular set S . The singularity occurs at time T . Let us now take ourselves in the time $T - \epsilon$, $\epsilon > 0$, in other words very close to the singularity in chronological terms. We cover the singular set S with the set C . As we are very close to singularity, the vorticity is already very large on S and its surroundings, in particular we choose the minimal covering set C in a way that the enstrophy density function is strictly positive in C

$$\phi = \vec{\omega} \cdot \vec{\omega} > 0 \quad (6)$$

which is always possible, as everything is still continuous. In addition, we choose the time parameter ϵ small enough such that at $T - \epsilon$ the enstrophy density function is larger on the singular set than anywhere else on the domain \mathbb{R}^3 .

2.1 The energy cascade

It is obvious that the velocity field blows up as well when the vorticity blows up, see [4], this can be seen from the definition of vorticity: the spatial derivatives of the velocity field become infinite by definition at the singularity. This can only happen if the velocity field itself becomes infinite at the singularity. As stated earlier we consider time $T - \epsilon$ with ϵ small enough to ensure that the velocity has a global maximum inside the covering set C .

As we move closer and closer to the singularity moment in terms of time, the process of forming of a singularity requires that energy density $f = \frac{1}{2} \vec{u} \cdot \vec{u}$ is transported from other parts of the system into the singular set, as otherwise the the divergence of velocity at the singularity would be impossible. It should be noted though that the velocity density is finite inside the set C until the moment the singularity occurs. Therefore if we have some minimal compact set C in the neighbourhood of S that is embedding the singular set, the net flux of energy density through the surface of C must be inwards and thus the flux integral must be negative. This means that

$$N(t) = \oint_{\partial C} \vec{F} \cdot \vec{C} \leq 0 \quad (7)$$

where \vec{F} represents the flux vector of energy density. Because the net flux satisfies $-\infty < N(t) \leq 0$ for all $t \geq T - \epsilon$, we can always represent the flux of energy density through the gradient field of the energy density in the following way.

By construction, we know that on the surface of the set C the gradient of energy

density points to the forming singularity, as we constructed the system in such a way that the global energy density maximum is on the singular set at time $t = T - \epsilon$. Because we know that $-\infty < N(t) \leq 0$ for all times $T - \epsilon \leq t < T$ we can always represent the net flux $N(t)$ with the choice of

$$N(t) = \oint_{\partial C} \sigma(t) \nabla f \cdot \vec{C} \leq 0 \quad (8)$$

where $\sigma(t) \geq 0$ is a scaling function that ensures that the net flux can always be represented through the gradient field of the energy density f . For the purpose of this study it is however sufficient to freeze the system at $t = T - \epsilon$ and study the system at that point in time. This means that instead of $\sigma(t)$ we take just some $\sigma > 0$ so that the net flux at that time instant can be represented through the gradient flux.

Let us now consider what the system does at such time close to singularity. The change in energy content of the covering set is given by

$$\frac{dK(t)}{dt} = \int_C \frac{\partial f}{\partial t} dV \quad (9)$$

This is so, because we have a *fixed control volume* C in space. So that the Reynolds transport theorem is reduced to the equation above. By definition, the time derivative of the energy content must be other hand related to the flux of energy density

$$\frac{dK(t)}{dt} = - \oint_{\partial C} \sigma(t) \nabla f \cdot \vec{C} \quad (10)$$

This tells that the only way that the integral of the energy density within the set C can increase is via the flux through the surface, inside the set. As the total kinetic energy is decreasing in the system, the increase of kinetic energy in some set must brought upon by transfer from other regions of the space. This is precisely what the previous equation states. Let us now use the divergence theorem.

$$\int_C \nabla \cdot \sigma(t) \nabla f dV = \oint_{\partial C} \sigma(t) \nabla f \cdot \vec{C} \quad (11)$$

As there is a global maximum for f inside the set C , we must have that

$$\Delta f \leq 0 \quad (12)$$

on the set C . Combining equations we have

$$\frac{dK(t)}{dt} = \int_C \frac{\partial f}{\partial t} dV = - \int_C \nabla \cdot \sigma(t) \nabla f dV \quad (13)$$

As the set C is arbitrary, we must have

$$\frac{\partial f}{\partial t} = -\sigma(t) \Delta f \quad (14)$$

So the energy cascade dynamics is also equivalent to backwards heat equation, which is known to behave badly. It can even blow up in finite time, given (im)proper conditions. Note that because we are close to singularity in chronological terms, the energy density has a global maximum inside the set C .

3 The Navier-Stokes equations and the singularity process

Let us now recall the Lamb formulation of Navier-Stokes equations

$$\frac{\partial \vec{u}}{\partial t} = -\vec{L} - \nabla(p + \frac{1}{2}\vec{u} \cdot \vec{u}) + \nu \Delta \vec{u} \quad (15)$$

Where the Lamb vector is defined as

$$\vec{L} = \vec{\omega} \times \vec{u} \quad (16)$$

If one now takes divergence, one has the divergence for the Lamb vector

$$\nabla \cdot \vec{L} = -\Delta(p + f) \quad (17)$$

So that we can reformulate the backwards heat equation for the energy cascade as

$$\frac{\partial f}{\partial t} = \sigma(t)(\nabla \cdot \vec{L} + \Delta p) \quad (18)$$

The Lamb vector divergence, or hydrodynamic charge, can be written also as

$$\nabla \cdot \vec{L} = \vec{u} \cdot \nabla \times \vec{\omega} - \omega \cdot \omega \quad (19)$$

Or in the form

$$\nabla \cdot \vec{L} = -\vec{u} \cdot \Delta \vec{u} - \omega \cdot \omega \quad (20)$$

see, [5], So that the energy cascade equation becomes

$$\frac{\partial f}{\partial t} = \sigma(t)(\Delta p - \vec{u} \cdot \Delta \vec{u} - \omega \cdot \omega) \quad (21)$$

Let us now take the inner product of velocity field with the Navier-Stokes equations

$$\vec{u} \cdot \frac{\partial \vec{u}}{\partial t} = -\vec{u} \cdot \vec{L} - \vec{u} \cdot \nabla(p + \frac{1}{2}\vec{u} \cdot \vec{u}) + \vec{u} \cdot \nu \Delta \vec{u} \quad (22)$$

This will simplify to

$$\frac{\partial f}{\partial t} = -\vec{u} \cdot \nabla(p + f) + \vec{u} \cdot \nu \Delta \vec{u} \quad (23)$$

As we know that the energy density is superharmonic on C , we deduce from the backwards heat equation

$$\frac{\partial f}{\partial t} \geq 0 \quad (24)$$

It must be so that

$$\vec{u} \cdot \nu \Delta \vec{u} \geq \vec{u} \cdot \nabla(p + f) \quad (25)$$

The well known result is, see [5]

$$\frac{1}{2} \vec{\omega} \cdot \vec{\omega} \leq -\vec{u} \cdot \Delta \vec{u} + \Delta f \quad (26)$$

Therefore we must have

$$\vec{u} \cdot \Delta \vec{u} \leq \Delta f - \frac{1}{2} \vec{\omega} \cdot \vec{\omega} \quad (27)$$

As we know that in the set C energy density is superharmonic, that is, $\Delta f \leq 0$ and that the enstrophy density is strictly positive in C we have

$$\vec{u} \cdot \Delta \vec{u} < 0 \quad (28)$$

Now the energy density evolution obtained from the Navier-Stokes equations must be equal to the energy density evolution deduced from the backwards heat equation in the set C . Therefore in the set C the the following equality must hold

$$-\vec{u} \cdot \nabla(p + f) + \vec{u} \cdot \nu \Delta \vec{u} = \sigma(t)(\Delta p - \vec{u} \cdot \Delta \vec{u} - \omega \cdot \omega) \quad (29)$$

Let us rearrange

$$\sigma(t) \Delta p = -\vec{u} \cdot \nabla(p + f) + (\nu + \sigma(t)) \vec{u} \cdot \Delta \vec{u} + \sigma(t) \vec{\omega} \cdot \vec{\omega} \quad (30)$$

So that we have

$$\Delta p = -\frac{\vec{u} \cdot \nabla(p + f)}{\sigma(t)} + \frac{(\nu + \sigma(t))}{\sigma(t)} \vec{u} \cdot \Delta \vec{u} + \vec{\omega} \cdot \vec{\omega} \quad (31)$$

We now consider the freezed system at time $t = T - \epsilon$ and we set $\sigma(t) = \sigma$. Let us consider now the maximal ratio of the following in C

$$\max_{\vec{x} \in C} \frac{|\vec{\omega} \cdot \vec{\omega} - \frac{\vec{u} \cdot \nabla(p+f)}{\sigma}|}{|\vec{u} \cdot \Delta \vec{u}|} \quad (32)$$

where $\vec{u} \cdot \Delta \vec{u} \neq 0$. We would not actually need the absolute value in the nominator as it is always positive in C following from the inequalities presented earlier. Let the maximal ratio be

$$\max_{\vec{x} \in C} \frac{|\vec{\omega} \cdot \vec{\omega} - \frac{\vec{u} \cdot \nabla(p+f)}{\sigma}|}{|\vec{u} \cdot \Delta \vec{u}|} = M \quad (33)$$

As $\vec{u} \cdot \Delta \vec{u} < 0$ is always satisfied in C , we can force the term Δp be to negative in C by choosing large enough value for the viscosity parameter ν . This forcing is satisfied if

$$\frac{\nu + \sigma}{\sigma} > M \quad (34)$$

So we choose naturally

$$\nu = \sigma M \quad (35)$$

Now what we have actually is that the pressure is superharmonic in C as we have $\Delta p \leq 0$. This means that the function $p + f$ is also superharmonic in C . Therefore we can deduce (using Weierstrass theorem and theory of superharmonic functions) that there is an interior maximum for the function $p + f$ and therefore for some $\vec{x} \in C$ the gradient must vanish

$$\nabla(p + f) = 0 \quad (36)$$

So at some point in the set C at time $t = T - \epsilon$ we must have that

$$\frac{\partial f}{\partial t} = \sigma M \vec{u} \cdot \Delta \vec{u} \quad (37)$$

Which implies that

$$\frac{\partial f}{\partial t} < 0 \quad (38)$$

Which is a contradiction as we had that in C at time $t = T - \epsilon$ we have

$$\frac{\partial f}{\partial t} \geq 0 \quad (39)$$

Which from we conclude that such energy cascade singularity process is impossible for the viscous, incompressible Navier-Stokes equations with such a choice of viscosity.

Because the Navier-Stokes equations exhibit scale invariance, applying a renormalization to the velocity field one deduces global regularity.

References

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